Two-sided mate choice problem

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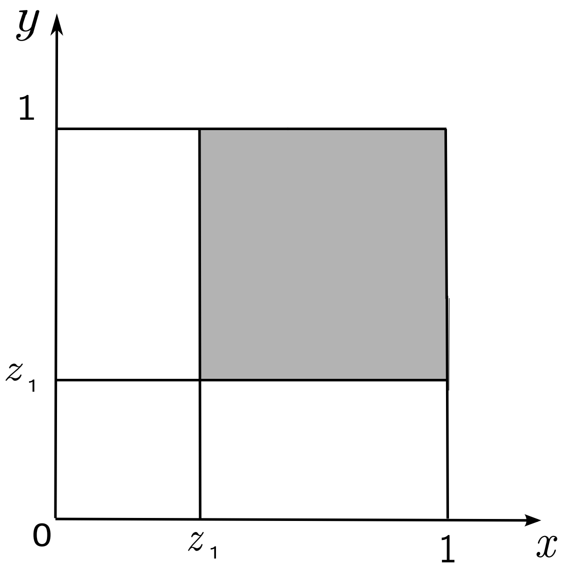
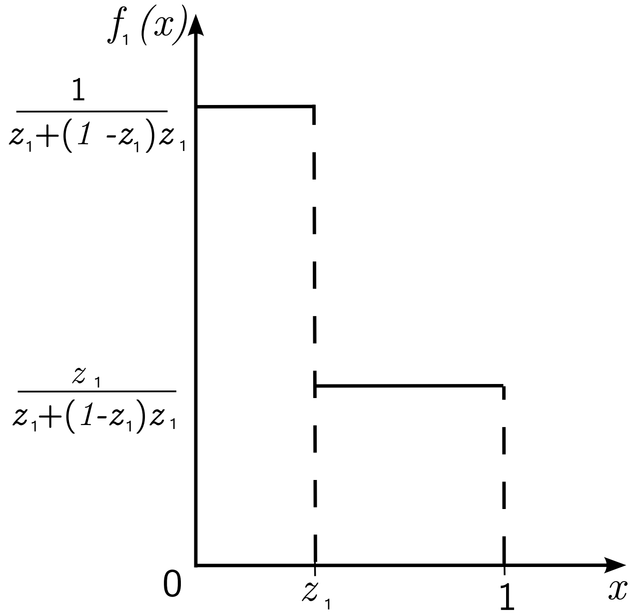
Two-sided mate choice problem

- Mating model, job search model
- The game with $n + 1$ stages
- $X = [0, 1]$ - females, $Y = [0, 1]$ - males. The quality of the members from each group x, y has uniform distribution
- If free individuals accept each other at the i-th stage they leave the game and each receives as a payoff the partner's quality.
- At the last stage $n + 1$ the individuals who don't create the pair receive zero
- Each player aims to maximize her/his expected payoff

Two-stage game

z_1 — the threshold of the acceptance at the first stage





$$f_1(x) = \begin{cases} \frac{1}{z_1+(1-z_1)z_1}, & x \in [0, z_1), \\ \frac{z_1}{z_1+(1-z_1)z_1}, & x \in [z_1, 1]. \end{cases}$$

The total number of individuals in each group at the second stage is equal to $z_1 + (1 - z_1)z_1$.

If a player doesn't mate at the first stage then his expected payoff (mean quality of the partner) at the second stage is

$$Ex_2 = \int_0^1 x f_1(x) dx = \frac{1 + z_1 - z_1^2}{2(2 - z_1)}.$$

$$z_1 = \frac{1 + z_1 - z_1^2}{2(2 - z_1)}.$$

It's solution $z_1 = (3 - \sqrt{5})/2 \approx 0.382$.

The game with $n+1$ stages

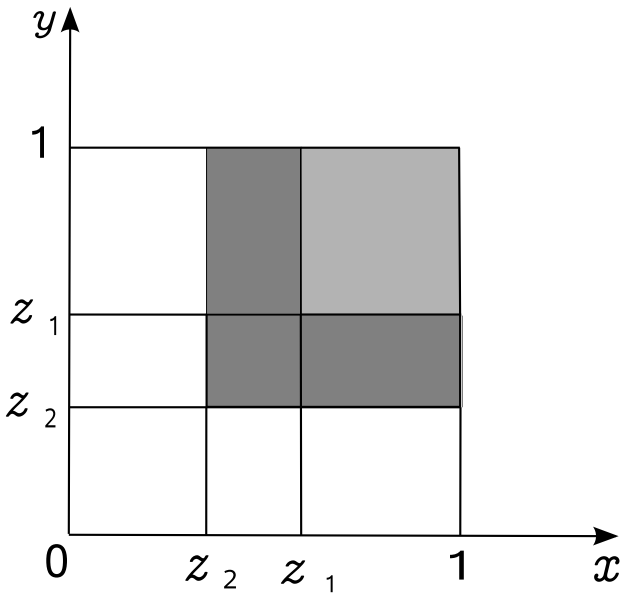
z_i — the threshold of the acceptance for the i-th stage ($i = 1, 2, ..., n$),
$0 < z_n \leq z_{n-1} \leq ... \leq z_1 \leq z_0 = 1$.

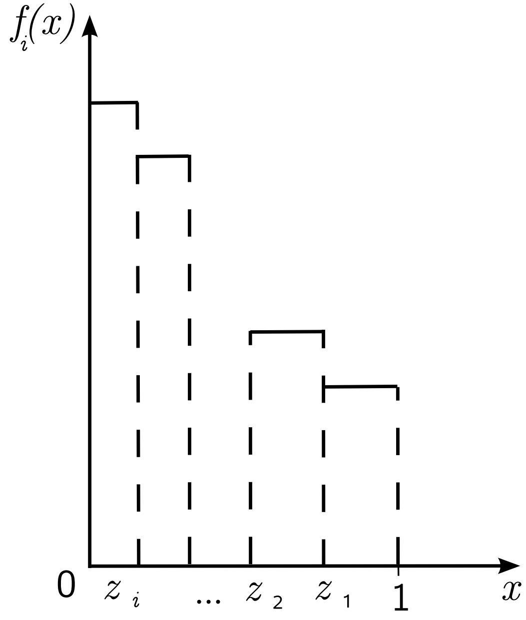
$N_0 = 1$;

$N_1 = z_1 + (1 - z_1)z_1$;

After the i-th stage obtain

$$N_i = 2z_i - \frac{z_i^2}{N_{i-1}},\ i = 1, ..., n. \tag{1}$$





After the i-th stage the distribution of players by quality has the density of the following form:

$$f_i(x) = \begin{cases} \frac{1}{N_i},\ 0 \leq x < z_i, \\ \prod_{j=k}^{i-1} \frac{z_{j+1}}{N_j} \frac{1}{N_i},\ z_{k+1} \leq x < z_k,\ k = i-1, ..., 1, \end{cases}$$

where $i = 1, ..., n$.

$v_i(x)$, $i = 1, ..., n$ the optimal expected payoff of the player after the i-th stage if he meets a partner with quality x

Hence,

$$v_n(x) = \max\{x, \int_0^1 y f_n(y) dy\}.$$

Then function $v_n(x)$ has the following form

$$v_n(x) = \begin{cases} z_n, \ 0 \le x < z_n, \\ x, \ z_n \le x \le 1. \end{cases}$$

the optimality equation after the i-th stage

$$v_i(x) = \max\{x, Ev_{i+1}(x_{i+1})\}$$

Theorem 1 *Nash equilibrium in the $(n+1)$-stage two-sided mate choice game is determined by the sequence of thresholds z_i, $i = 1, ..., n$, which satisfy the recurrence relation*

$$z_1 = \frac{1}{a_1}\left(1 - \sqrt{1 - a_1^2}\right), \quad z_i = a_i z_{i-1}, \, i = 2, ..., n,$$

where coefficients a_i satisfy the equations

$$a_i = \frac{2}{3 - a_{i+1}^2}, \, i = 1, ..., n-1, \tag{2}$$

and $a_n = 2/3$.

Thresholds in the two-sided (z_i) and in the one-sided $(\bar{z}_i)$ problem for $n = 10$.

i	1	2	3	4	5	6	7	8	9	10
a_i	0.940	0.934	0.927	0.918	0.907	0.891	0.870	0.837	0.782	0.666
z_i	0.702	0.656	0.608	0.559	0.507	0.452	0.398	0.329	0.308	0.205
$\bar{z}_i$	0.861	0.850	0.836	0.820	0.800	0.775	0.742	0.695	0.625	0.5

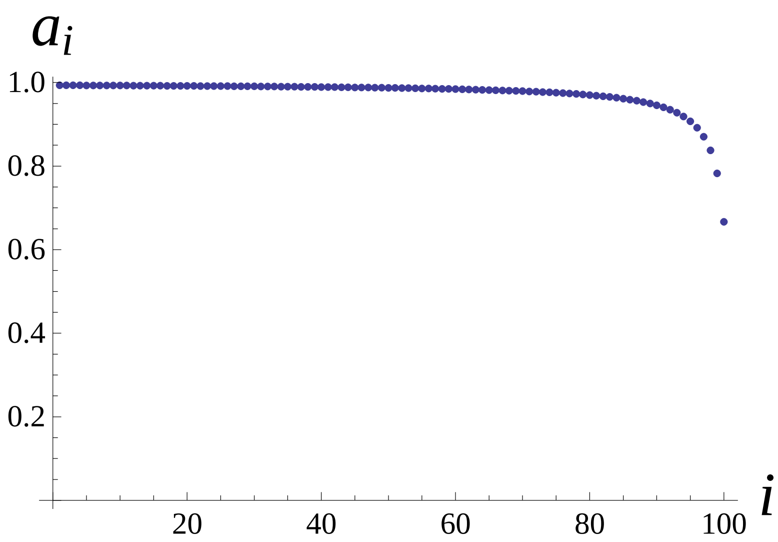


Figure 1. $a_i, n = 100$

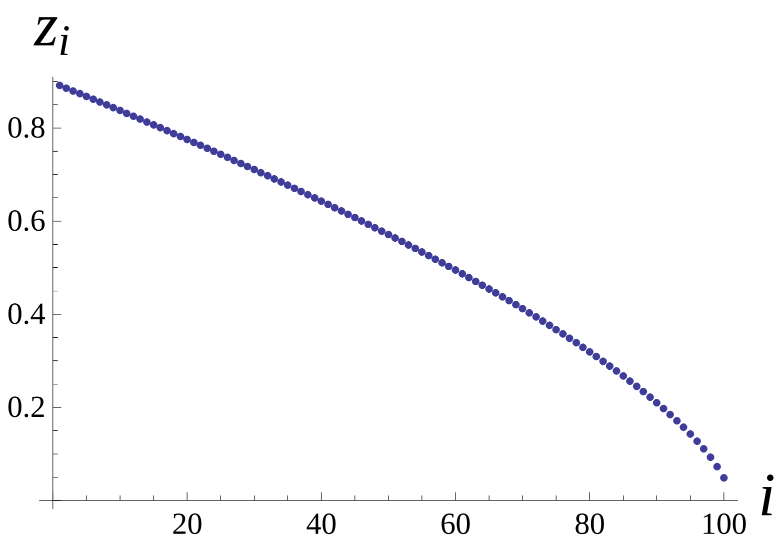


Figure 2. $z_i, n = 100$

For $n \to \infty \quad a' = a - \frac{2}{3-a^2}$, $a(n) = \frac{2}{3}$.

$$\frac{1}{9}\left(\frac{6}{a-1} + 8\ln\frac{1-a}{a+2}\right) = t + \frac{1}{9}\left(-18 - 9n + \ln\frac{8}{3^9}\right).$$

This equation estimates a_i from below.

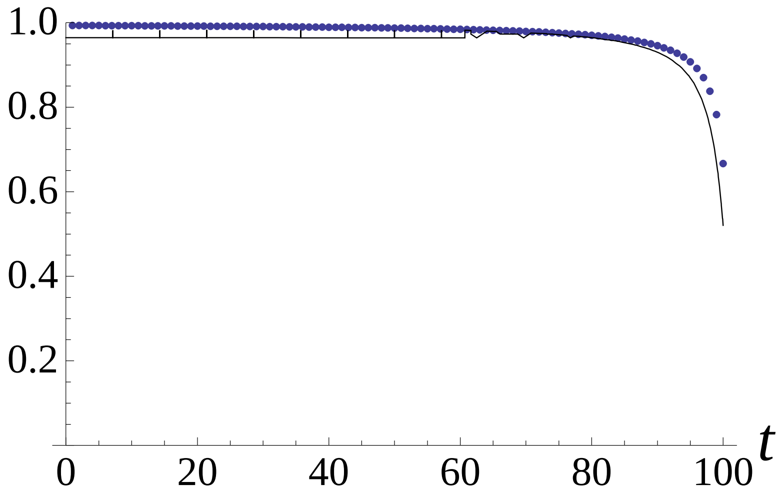


Figure 3. Blue — a_i, Black — $a(t)$, $n = 100$

$$a_i \geq 1 - \frac{2}{3(n-i+2)}$$

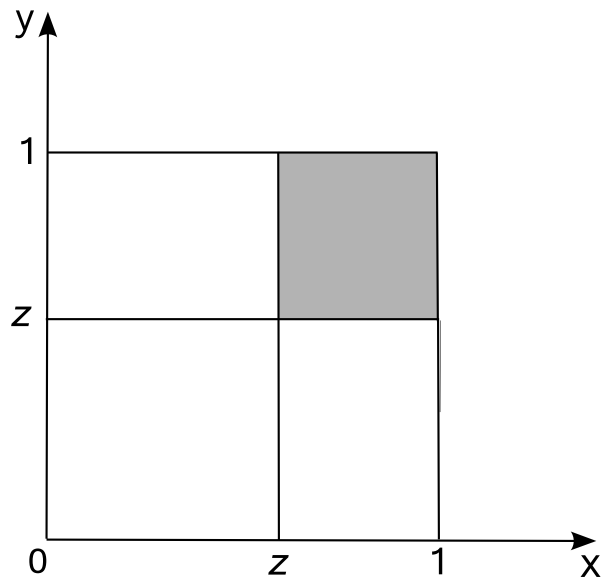
Two-sided mate choice problem with arriving flow

- The game with $n+1$ stages
- $X=[0,1]$ - females, $Y=[0,1]$ - males. The quality of the members from each group x, y has uniform distribution
- If free individuals accept each other at the i-th stage they leave the game and each receives as a payoff the partner's quality.
- At the last stage $n+1$ the individuals who don't create the pair receive zero
- There is a stream Δ_i of the new individuals at each stage
- Each player aims to maximize her/his expected payoff

Two-stage game

z_1 — the threshold of the acceptance at the first stage

$\Delta_1 = (1 - z_1)^2\alpha$, parameter α is birth rate, $\alpha \geq 0$.



The total number of individuals in each group at the second stage is equal to $N_1 = z_1 + (1 - z_1)z_1 + \Delta_1$.

The density of the distribution of the qualities at the second stage is following

$$f_1(x) = \begin{cases} \frac{1+\Delta_1}{N_1}, & 0 \le x < z_1; \\ \\ \frac{z_1+\Delta_1}{N_1}, & z_1 \le x \le 1. \end{cases}$$

As before we find the optimal value z_1 from the condition

$$z_1 = \int_0^1 x f_1(x) dx.$$

The equation for optimal threshold z_1

$$(1 - z_1)^2 \alpha = \frac{z_1(1 - 3z_1 + z_1^2)}{2z_1 - 1}.$$

Optimal thresholds z_1 in the model with arrival for various α

α	0	0.1	0.5	1	5	10	100	1000
z_1	0.382	0.391	0.414	0.430	0.469	0.481	0.498	0.5

Table presents the numerical results for the optimal values z_1 and z_2 for various α (three-stage game).

α	0	0.1	0.5	1	5	10	100	1000
z_1	0.482	0.498	0.529	0.548	0.590	0.603	0.622	0.625
z_2	0.322	0.346	0.391	0.416	0.467	0.480	0.498	0.5

The case of $n+1$ stages

z_i — the threshold of the acceptance for the i-th stage ($i = 1, 2, ..., n$)

After the i-th stage obtain

$N_1 = z_1 + (1 - z_1)z_1 + \Delta_1;$

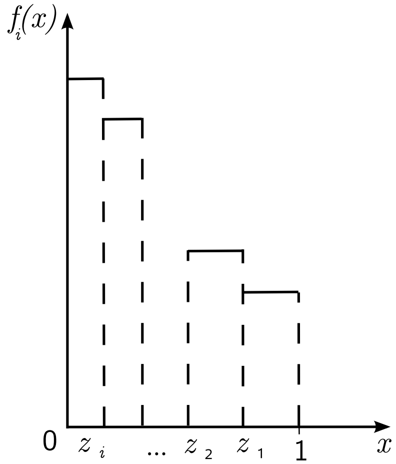
$N_2 = z_2(1+\Delta_1) + \frac{(z_1 - z_2)(1+\Delta_1)z_2(1+\Delta_1)}{N_1} + \frac{(1-z_1)(z_1+\Delta_1)z_2(1+\Delta_1)}{N_1} + \Delta_2;$

$$N_i = z_i(1 + \sum_{j=1}^{i-1} \Delta_j)\left[2 - \frac{z_i(1+\sum_{j=1}^{i-1} \Delta_j)}{N_{i-1}}\right] + \Delta_i, \qquad (3)$$

and Δ_i is determined by $\Delta_i = \alpha \sum_{j=1}^{i} \bar{N}_j$, where $\bar{N}_j$ is the number of individuals who form the pair at the j-th stage.

The density of the distribution at the stage $i+1$ $(i = 1, ..., n)$

$$f_i(x) = \begin{cases} \dfrac{1+\sum\limits_{j=1}^{i}\Delta_j}{N_i}, & 0 \le x < z_i; \\[2ex] \dfrac{(1+\sum\limits_{j=1}^{i-1}\Delta_j)\frac{z_i(1+\sum\limits_{j=1}^{i-1}\Delta_j)}{N_{i-1}}+\Delta_i}{N_i}, & z_i \le x < z_{i-1}; \\[2ex] \dfrac{\left[\ldots\left[[z_1+\Delta_1]\frac{z_2(1+\Delta_1)}{N_1}+\Delta_2\right]\frac{z_3(1+\sum\limits_{j=1}^{2}\Delta_j)}{N_2}+\ldots+\Delta_{i-1}\right]\frac{z_i(1+\sum\limits_{j=1}^{i-1}\Delta_j)}{N_{i-1}}+\Delta_i}{N_i}, & z_1 \le x \le 1. \end{cases} \quad (4)$$



Let $v_i(x), i = 1, ..., n$ be the optimal expected payoff of a player from population Y if he meets a partner with quality x.

$$v_n(x) = \max\{x, \int_0^1 y f_n(y) dy\},$$

$$v_i(x) = \max\{x, Ev_{i+1}(x_{i+1})\}, i = 1, ..., n-1.$$

Theorem 2 *Nash equilibrium in the $(n+1)$-stage two-sided mate choice game with arriving individuals is determined by the sequence of thresholds z_i, $i = 1, ..., n$, which satisfies the recurent equations*

$$\begin{cases} z_n &= \int_0^1 x f_n(y) dy; \\ z_i &= \int_0^{z_{i+1}} z_{i+1} f_i(y) dy + \int_{z_{i+1}}^1 y f_i(y) dy, i = 1, 2, ..., n-1, \end{cases} \tag{5}$$

where $f_i(x)$ satisfy (4).

Let us find the asymptotic behavior of the optimal thresholds as $\alpha \to \infty$.

Lemma 1 *For all* $i = 1, ..., n$ $\lim_{\alpha \to \infty} f_i(x) = 1$.

Theorem 2 with Lemma 1 gives immediately the Corollary.

Corollary 1 *As* $\alpha \to \infty$ *the optimal thresholds* z_i *(*$i = 1, ..., n$*) satisfy the recurrent formulas*

$$z_i = \frac{1 + z_{i+1}^2}{2}, i = 1, ..., n-1; z_n = 1/2.$$

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