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PERFORMANCE ANALYSIS OF BRIDGE MONTE-CARLO ESTIMATOR

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The overflow probability is an important QoS (Quality of Service) parameter. In this paper, we analyze the performance of Bridge Monte-Carlo (BMC), an interesting approach for the estimation of the overflow probability for queueing systems fed by a Gaussian input process.

Key words: Gaussian queue, fractional Brownian motion, overflow probability, estimation.

О. В. Лукашенко, Е. В. Морозов, М. Пагано. АНАЛИЗ ЭФФЕКТИВНОСТИ ВМС-ОЦЕНКИ

Вероятность переполнения является важным показателем качества обслуживания. В данной статье мы изучаем качественные свойства ВМС-оценки этой вероятности для различных гауссовских входных процессов.

Ключевые слова: гауссовская очередь, дробное броуновское движение, вероятность переполнения, оценивание.

INTRODUCTION

We consider a single server queueing system with constant service rate C fed by a Gaussian input, which is defined as follows:

$$A_t = mt + X_t, \quad (1)$$

where constant $m > 0$ and $\{X_t, t \in T\}$ with $T = \{\mathbb{Z}_+\}$ (or $T = \{\mathbb{R}_+\}$) is a centered Gaussian process with stationary increments, which describes random fluctuations of the input around its linearly increasing mean. To guarantee stability of such a system we assume that $m < C$. Let us denote $v_t := \mathbb{D}X_t$ – the variance of X_t . Then the covariance function has the following expression:

$$\Gamma_{s,t} = \frac{1}{2} (v_t + v_s - v_{|t-s|}). \quad (2)$$

The stationary overflow probability (i.e., the probability that stationary workload Q exceeds some threshold level B) has the following representation [12]:

$$\begin{aligned} \mathbb{P}_{\text{overflow}} &:= \mathbb{P}(Q \geq B) \\ &= \mathbb{P}\left(\sup_{t \in T} (A_t - Ct) \geq B\right) \\ &= \mathbb{P}\left(\sup_{t \in T} (X_t - \varphi_t) \geq 0\right), \end{aligned} \quad (3)$$

where $\varphi_t := B + rt$, $r := C - m > 0$.

We consider the following important cases of Gaussian inputs:

1. Fractional Brownian Motion (FBM). In this case $v_t = t^{2H}$, with Hurst parameter $H \in (0, 1)$ (in the teletraffic framework usually $H \in (0.5, 1)$), corresponding to traffic processes with long range

dependence). It has been shown in [13] that FBM arises as the scaled limit process when the cumulative workload is a superposition of on-off sources with mutually independent heavy-tailed on and/or off periods.

2. Sum of two independent FBMs with $v_t = t^{2H_1} + t^{2H_2}$. The use of this model is also motivated by the fundamental result in [13] in case of heterogeneous on-off sources.

3. Integrated Ornstein-Uhlenbeck process (IOU) with $v_t = t + e^{-t} - 1$. IOU is the Gaussian counterpart of the well-known Anick-Mitra-Sondi fluid model [1], and its relevance is further motivated in [8].

ASYMPTOTIC REGIMES

There are no explicit expressions for (3) in case of general Gaussian input (there are some results for specific simple cases like standard Brownian motion). Therefore researches were concentrated on asymptotic analysis and simulation technique in different regimes which are described below.

Large buffer regime

In this regime the overflow probability

$$\mathbb{P}_B = \mathbb{P}(Q \geq B)$$

is analyzed for large B . The following logarithmic asymptotic result has been found in [3]:

$$\log \mathbb{P}_B \sim - \inf_{t \geq 0} \frac{V^2(t)}{2}, \text{ as } B \rightarrow \infty, \quad (4)$$

where $f \sim g$ means $f/g \rightarrow 1$ and

$$V(t) = \frac{B + rt}{\sqrt{v_t}}.$$

Expression (4) means that for sufficiently large values of B

$$\mathbb{P}_B \approx \exp \left(- \inf_{t \geq 0} \frac{V^2(t)}{2} \right). \quad (5)$$

The so-called most-likely time τ of the overflow is the optimizing argument in (4) and (5). For FBM input, time τ has the following explicit form

$$\tau = \frac{H}{1-H} \cdot \frac{B}{r}, \quad (6)$$

implying

$$V(\tau) = \left(\frac{B}{1-H} \right)^{1-H} \left(\frac{r}{H} \right)^H. \quad (7)$$

Calculation of exact asymptotics (which are more informative than log asymptotics) is typically much more difficult problem depending on the Gaussian component X of the input. We refer to [7, 10, 11] where such results can be found.

Many sources regime

Often in a large network the input to a station is typically a superposition of a large number n of the streams generated by the i.i.d. sources. This observation leads to analysis of the so-called many sources regime where the input to a station has a form $A_t = mnt + \sum_{i=1}^n X_t^i$, with the i.i.d. centered Gaussian processes $\{X^i\}$ (with stationary increments), and the threshold and capacity are scaled accordingly, i. e., $B = nb$ and $C = nc$ where parameter $b > 0$ and c corresponds to capacity of a single station. Let now be $r := c - m > 0$ and $\varphi_t = b + rt$. Then the overflow probability in the many-source regime becomes:

$$\begin{aligned} \mathbb{P}_n &:= \mathbb{P} \left(\sup_{t \in T} \left(\sum_{i=1}^n X_t^i - nrt \right) \geq nb \right) \\ &= \mathbb{P} \left(\sup_{t \in T} \left(\sum_{i=1}^n X_t^i - n\varphi_t \right) \geq 0 \right) \\ &= {}_d \mathbb{P} \left(\sup_{t \in T} \left(X_t^{(n)} - \varphi_t \right) \geq 0 \right), \end{aligned}$$

where $X_t^{(n)} := \sqrt{1/n} X_t$ (X_t denotes a generic element of X_t^i). Note that the last expression corresponds to equation (3) with the Gaussian input component $X_t^{(n)}$.

There are several asymptotic results for the overflow probability. Most of them claim that, under mild conditions, such a probability decays exponentially fast in n . The following results has been proved in [2]:

$$- \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_n = \inf_{t \geq 0} \frac{V^2(t)}{2}, \quad (8)$$

where

$$V(t) = \frac{b + rt}{\sqrt{v_t}}.$$

Expression (8) means that for n sufficiently large

$$\mathbb{P}_n \approx \exp \left(-n \inf_{t \geq 0} \frac{V^2(t)}{2} \right). \quad (9)$$

As in a large buffer regime, the optimizing argument τ in (8) is called the most-likely time of the overflow. Result (8) gives only logarithmic

asymptotics. In discrete time the following exact large deviation (LD) asymptotic holds [9]:

$$\mathbb{P}_n \sim \Phi(V(\tau)\sqrt{n}), \quad n \rightarrow \infty, \quad (10)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy.$$

For other exact asymptotics we refer to [4].

BMC ESTIMATOR

Bridge Monte-Carlo (BMC) is a new approach to estimation of the overflow probability in a queueing system with Gaussian input.

Originally proposed by some of the authors in [5], BMC is based on the idea of expressing the overflow probability as the expectation of a function of the *Bridge* $Y := \{Y_t\}$ of the Gaussian input process X , i.e., the process obtained by conditioning X to reach a certain level at some prefixed time \bar{t} :

$$Y_t = X_t - \psi_t X_{\bar{t}}, \quad (11)$$

where function ψ_t is expressed via covariance function Γ as

$$\psi_t := \frac{\Gamma_{t,\bar{t}}}{\Gamma_{\bar{t},\bar{t}}}.$$

Because the variance of the input is increasing function of t in all models we consider in the paper, it is easy to see that $\psi_t > 0$ for all $t \in T$. Moreover, we note that the process Y is independent of $X_{\bar{t}}$ since

$$\mathbb{E}[X_{\bar{t}} Y_t] = \Gamma_{\bar{t},t} - \frac{\Gamma_{t,\bar{t}}}{\Gamma_{\bar{t},\bar{t}}} \Gamma_{\bar{t},\bar{t}} = 0.$$

Further, we have

$$\begin{aligned} \mathbb{P}_{\text{overflow}} &:= \mathbb{P}\left(\sup_{t \in T} (X_t - \varphi_t) \geq 0\right) \\ &= \mathbb{P}\left(\sup_{t \in T} (Y_t + \psi_t X_{\bar{t}} - \varphi_t) \geq 0\right) \\ &= \mathbb{P}\left(\inf_{t \in T} (\varphi_t - Y_t - \psi_t X_{\bar{t}}) \leq 0\right). \end{aligned}$$

Consider two events:

$$\begin{aligned} A &= \left\{ \inf_{s \in T} (\varphi_s - Y_s - \psi_s X_{\bar{t}}) \leq 0 \right\}, \\ B &= \left\{ \inf_{t \in T} \psi_t^{-1} [\varphi_t - Y_t] \leq X_{\bar{t}} \right\}. \end{aligned}$$

Fix any $\omega \in A$ and let $s^* = \operatorname{argmin}(\varphi_s - Y_s - \psi_s X_{\bar{t}})$. Note that the event A is not empty since

$$\varphi_{\bar{t}} - Y_{\bar{t}} - \psi_{\bar{t}} X_{\bar{t}} = 0.$$

Then

$$\varphi_{s^*} - Y_{s^*}(\omega) - \psi_{s^*} X_{\bar{t}}(\omega) \leq 0.$$

Thus, the following inequality holds

$$\inf_{t \in T} \psi_t^{-1} [\varphi_t - Y_t(\omega)] \leq \psi_{s^*}^{-1} [\varphi_{s^*} - Y_{s^*}(\omega)] \leq X_{\bar{t}}(\omega).$$

That is $\omega \in B$, and hence $A \subseteq B$. Similarly, we can check that $B \subseteq A$. It means that $A = B$. Denote

$$\bar{Y} := \inf_{t \in T} \frac{\varphi_t - Y_t}{\psi_t}. \quad (12)$$

Recall that

$$X_t = N(0, \Gamma_{t,t}) =_d \sqrt{\Gamma_{\bar{t},\bar{t}}} N(0, 1),$$

where $=_d$ stands for stochastic equivalence. Then, the overflow probability can be rewritten as follows

$$\begin{aligned} \mathbb{P}_{\text{overflow}} &= \mathbb{P}(\bar{Y} \leq X_{\bar{t}}) = \int_R \mathbb{P}(X_{\bar{t}} \geq u) \mathbb{P}(\bar{Y} \in du) \\ &= \int_R \mathbb{P}\left(N(0, 1) \geq \frac{u}{\sqrt{\Gamma_{\bar{t},\bar{t}}}}\right) \mathbb{P}(\bar{Y} \in du) \\ &= \mathbb{E}\left[\Phi\left(\frac{\bar{Y}}{\sqrt{\Gamma_{\bar{t},\bar{t}}}}\right)\right], \end{aligned}$$

where independence \bar{Y} and $X_{\bar{t}}$ is used. Given an i.i.d sequence $\{\bar{Y}^{(i)}, i = 1, \dots, N\}$ distributed as \bar{Y} , the estimator of $\mathbb{P}_{\text{overflow}}$ is defined as follows:

$$\hat{\mathbb{P}}_{\text{overflow}} := \hat{\mathbb{P}}_{\text{overflow}}(N) = \frac{1}{N} \sum_{i=1}^N \Phi\left(\frac{\bar{Y}^{(i)}}{\sqrt{\Gamma_{\bar{t},\bar{t}}}}\right).$$

In spite of the fact that the BMC estimator is not asymptotically efficient, its variance is lower than for the single-twist Importance Sampling (which is comparable in the terms of computational complexity) [6]. Moreover, the approach using BMC estimator is extremely flexible since it does not rely on a change of measure. Furthermore, to apply this estimator the knowledge of the correlation structure of the incoming traffic is only required. (As we mentioned above the assumption of the Gaussianity of X_t is typically fulfilled when a lot of flows are multiplexed together.) Although the choice of \bar{t} is arbitrary, in the following we will always assume that $\bar{t} = \tau$, i.e. as the conditioning point we will consider the most-likely time of the overflow. For a wide range of values of the queue parameters, the minimum in (12) is almost always attained near the most-likely time and does not vary significantly. Let us denote

$$G(t) := \frac{\varphi_t - Y_t}{\psi_t}.$$

and note that $G(\bar{t}) = \varphi_{\bar{t}}$ is deterministic.

Assume that $\bar{Y}^{(i)} \in [\varphi_{\tau} - h, \varphi_{\tau}]$, where evidently the span h depends on the samples $\{Y^{(i)}, i = 1, \dots, N\}$. Then by the monotonicity of the tail distribution Φ ,

$$\Phi\left(\frac{\varphi_{\tau}}{\sqrt{\Gamma_{\tau,\tau}}}\right) \leq \hat{\mathbb{P}}_{\text{overflow}} \leq \Phi\left(\frac{\varphi_{\tau} - h}{\sqrt{\Gamma_{\tau,\tau}}}\right). \quad (13)$$

Consider the difference

$$\Delta(h) := \Phi\left(\frac{\varphi_{\tau} - h}{\sqrt{\Gamma_{\tau,\tau}}}\right) - \Phi\left(\frac{\varphi_{\tau}}{\sqrt{\Gamma_{\tau,\tau}}}\right),$$

which can be approximated as

$$\begin{aligned} \Delta(h) &\approx -\Phi'\left(\frac{\varphi_{\tau}}{\sqrt{\Gamma_{\tau,\tau}}}\right) \frac{h}{\sqrt{\Gamma_{\tau,\tau}}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} \frac{h}{\sqrt{\Gamma_{\tau,\tau}}}, \end{aligned} \quad (14)$$

where $Z = \frac{\varphi_{\tau}}{\sqrt{\Gamma_{\tau,\tau}}}$. Actually if the distance Δ between lower and upper bound in (13) is not too large, we can estimate the accuracy of approximation (10). We note that $V(\tau)\sqrt{n} = \frac{\varphi_{\tau}}{\sqrt{\Gamma_{\tau,\tau}}}$, so expression (10) indeed gives only lower bound of $\hat{\mathbb{P}}_{\text{overflow}}$. Below we verify the accuracy of approximation (10) by simulation.

SIMULATION RESULTS

In this section, a few numerical results are presented which demonstrate the properties and accuracy of the BMC estimator.

We first show the accuracy of the BMC estimator for the different input processes, by comparing the simulation results with the known asymptotics (both for large buffer regime and for many sources regime).

Then we investigate the properties of the BMC estimator from an analytical point of view, taking into account the dependence of the conditional overflow probability from the simulated sample paths of the input process in the case of FBM traffic.

Comparison with asymptotic results

The first set of simulations compares the estimates of BMC with the asymptotic expressions recalled above.

Figures 1–3 refer to many sources regime for different input processes: FBM, sum of independent FBMs and IOU, respectively. In all cases the estimation of the overflow probability

uses $N = 10^6$ sample paths and is compared with the exact asymptotic given by (10). The following parameters are used in simulation: $r = 0.1$; $b = 0.3$; $H = H_1 = 0.8$; $H_2 = 0.6$. As figures show, a good consistency between theoretical values and simulation results are obtained over a wide range of the overflow probability values.

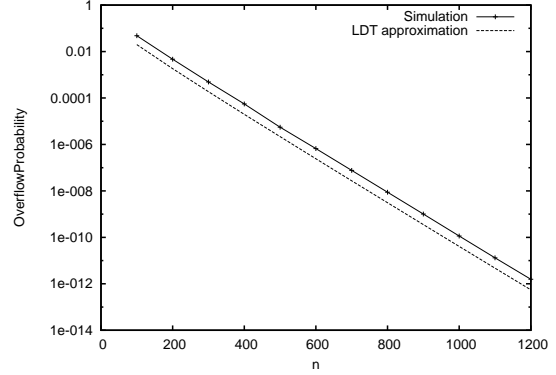


Fig. 1. Simulation vs. asymptotic (10): FBM

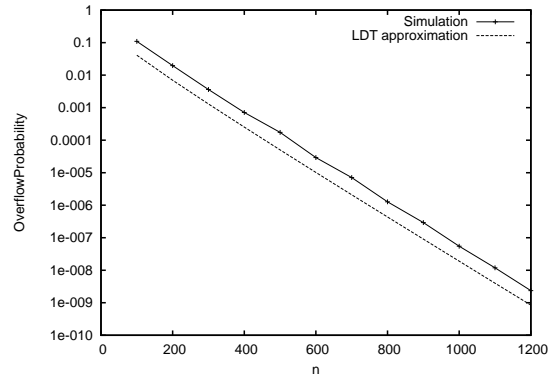


Fig. 2. Simulation vs. asymptotic (10): the sum of two independent FBMs

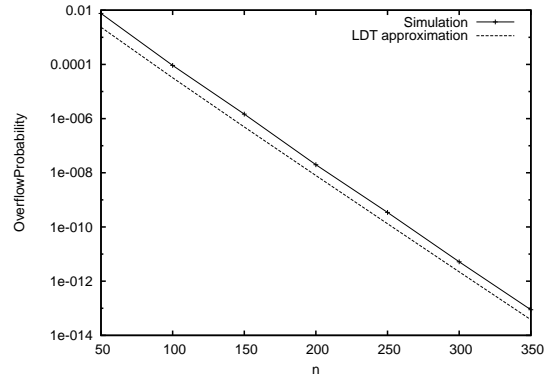


Fig. 3. Simulation vs. asymptotic (10): IOU

Moreover, for FBM input Figure 4 shows the behavior of the relative error of the BCM estimator (defined as the ratio between the empirical standard deviation and the corresponding probability). Although the relative error is not bounded (indeed, BMC is not even asymptotically efficient [6]), it grows slowly, and for the overflow probabilities of the order of 10^{-12} (compare the values in figures 1 and 4) is still less than 1 %.

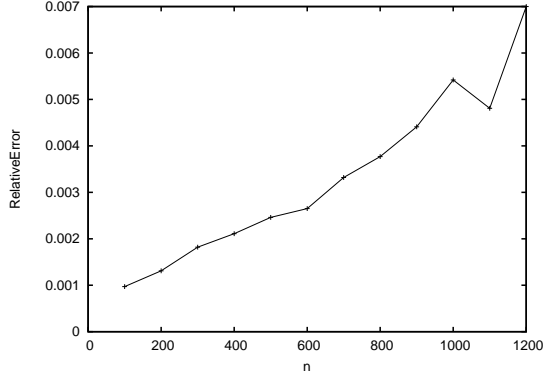


Fig. 4. Relative Error for FBM

Finally, figure 5 refers to the large buffer regime and compares the LD bound (5) with the simulation results in the case of a single FBM (with $H = 0.8$ as before) process. In this case B goes from 10 to 100 and $r = 1$, considering $N = 10^4$ sample paths (the choice is motivated by the relative high values of the simulated probability).

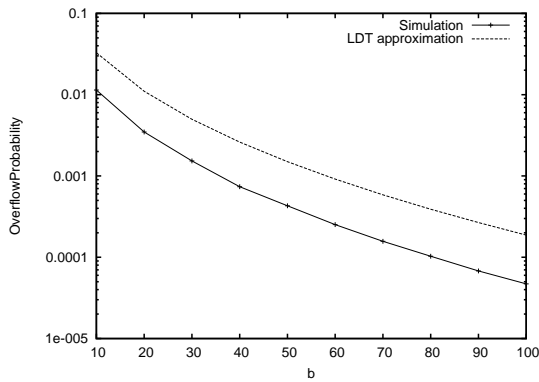


Fig. 5. Simulation vs. asymptotic (5) for FBM

Performance analysis of BMC

The second set of simulations aimed at checking the variability of h , and hence of Δ in

(14), in order to understand the goodness of the asymptotic approximation (10).

The tests are performed considering a single FBM flow in the many sources regime and using the same parameters as in previous section.

To give a visual idea of the variability of $\bar{Y}^{(i)}$, figure 6 compares its first 1000 samples with the theoretical upper bound $G(\tau) = \varphi_\tau$ for $n = 500$ FBM sources.

As highlighted in the figure, in this example $h \approx 0.298$ is not significantly lower than $\varphi_\tau = 1.5$, confirming the goodness of the LD approximation (10).

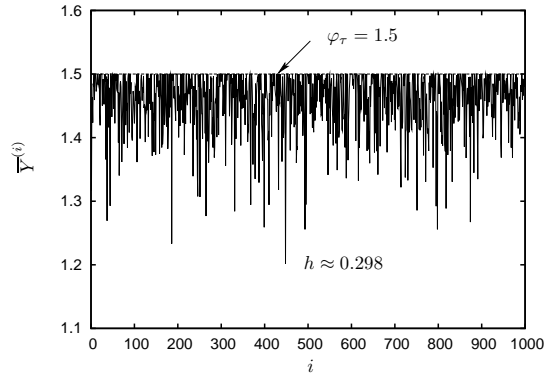


Fig. 6. Simulation results for $\bar{Y}^{(i)}$

To better understand the variability of \bar{Y} , figures 7 – 10 shows the empirical distribution of \bar{Y} for different values of n . As expected, for large values of n , \bar{Y} is concentrated near $G(\tau) = 1.5$, and this fact gives a formal motivation for the analysis of Δ in (14).

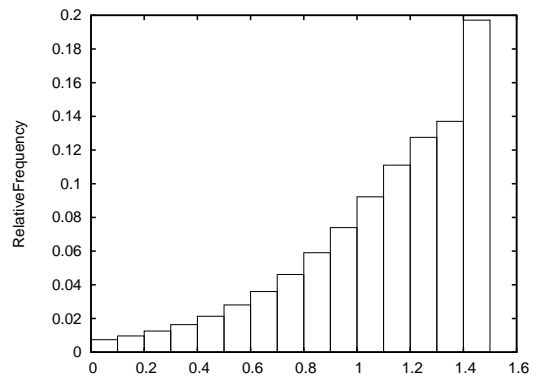


Fig. 7. Histogram of the distribution of \bar{Y} ($n = 50$)

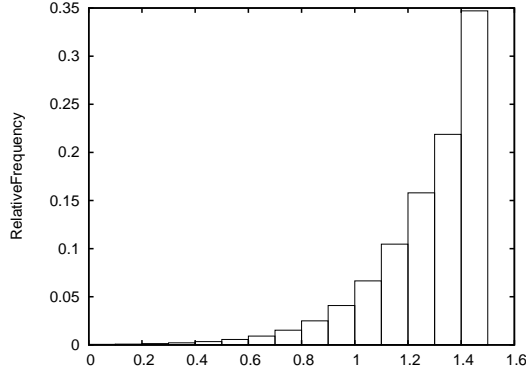


Fig. 8. Histogram of the distribution of \bar{Y} ($n = 100$)

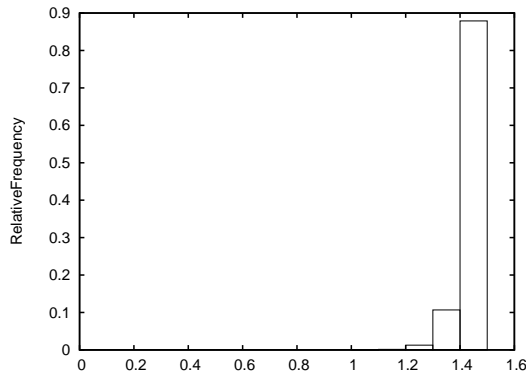


Fig. 9. Histogram of the distribution of \bar{Y} ($n = 500$)

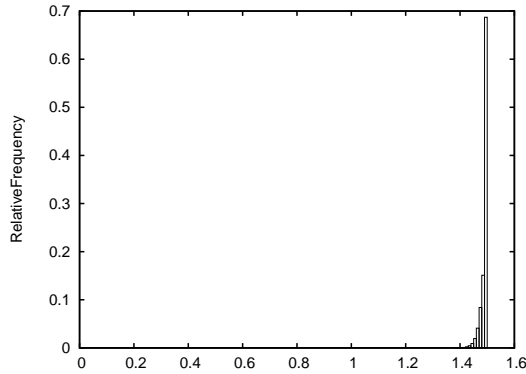


Fig. 10. Histogram of the distribution of \bar{Y} ($n = 2000$)

For sake of completeness, figure 11 shows the variation of $\bar{Y}^{(i)}$ for FBM in the large buffer regime (buffer size $b = 2000$) and the following values of the system parameters: $H = 0.8$; $r = 1$; $N = 10^3$. In this example $h \approx 658.941$, but the ratio $\frac{h}{\sqrt{\Gamma_{\tau, \tau}}} \approx 0.497$. Thus, inspite of that h is rather large, the increment of the argument of

function Φ in (13) is $\frac{h}{\sqrt{\Gamma_{\tau, \tau}}}$ and comparably small. It shows that approximation (14) can be applied.

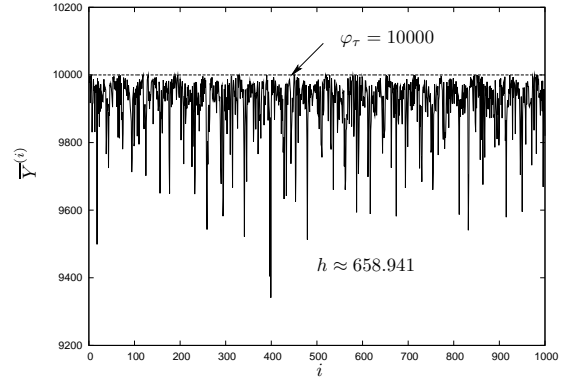


Fig. 11. Simulation results for $\bar{Y}^{(i)}$

CONCLUSIONS

In this paper, we have analyzed the main properties of BMC estimator, a simulation approach that exploits the Gaussian nature of the input process and relies on the properties of Bridges.

Several sets of simulations were carried out in order to compare the estimations with well-known asymptotic bounds for different input processes and in different working conditions, considering large buffers as well as the superposition of many i.i.d. sources.

Focusing on the latter scenario, we investigated the empirical distribution of the estimates and the dependence of the conditional overflow probability from the simulated sample paths of the bridge process, in order to understand the applicability of asymptotic results. The simulations highlighted that the shape of the histograms strongly depends on the number of multiplexed sources, confirming the well-known heuristic that rare event happens in the more likely way.

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