On connection between permutation complexity and factor complexity of infinite words

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Basic definitions

An infinite word over the alphabet Σ is a word of the form $\omega = \omega_1\omega_2\omega_3\ldots$, where $\omega_i \in \Sigma$.

Subword

A word u is called a *subword* or *factor* of length n of an infinite word ω if $u = \omega_{i+1}\ldots\omega_{i+n}$ for some $i \geq 0$.

Factor complexity

The *factor complexity* $C(n)$ of word ω is the number of its distinct subwords of length n.

Infinite permutations

Definition

An ordered triple $\delta = \langle \mathbb{N}, <_\delta, < \rangle$, where $<_\delta$ is some order on the set $\mathbb{N}$ and $<$ is the natural order on $\mathbb{N}$, is called an *infinite permutation*.

Thus, an infinite permutation is a linear order on the set of positive integer numbers.

Example

1. Let $a_n = (-1/2)^n$. Then a_n defines the order $<_{\delta_1}$: $i <_{\delta_1} j \Leftrightarrow a_i < a_j$.
2. Let $b_n = 1000 + (-1/n)^n$. Then b_n defines the order $<_{\delta_2}$: $i <_{\delta_2} j \Leftrightarrow b_i < b_j$.

We have $\delta_1 = \delta_2$.

Subpermutations

We will use the definition of a *finite permutation* x of length n as a linear order on $\{1, 2, \ldots, n\}$ which can be different from the natural one. In what follows we will write $x = x_1 x_2 \ldots x_n$, if $\{x_1, x_2, \ldots, x_n\}$ is a permutation of numbers from the set $\{1, 2, \ldots, n\}$ such that $x_i < x_j$ if and only if $i <_x j$.

Definition

$\delta[m, m+n-1] = x_1 x_2 \ldots x_n$ is the finite permutation of length n such that $x_i < x_j$ if and only if $m+i-1 <_\delta m+j-1$.

Definition

A finite permutation π is a *subpermutation* of length n of an infinite permutation δ if $\pi = \delta[i, i+n-1]$ for some $i > 0$.

Factor complexity of infinite permtutations

Definition

The *factor complexity* $\lambda(n)$ of an infinite permutation δ is the number of its distinct subpermutations of length n.

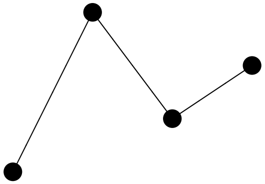
Example of subpermutation

Let $a_n = (-1/2)^n$. Then a_n defines the order $<_\delta$:
$i <_\delta j \Leftrightarrow a_i < a_j$.

Permutation $\delta[1,4]$

Since $a_1 = -1/2$, $a_2 = 1/4$, $a_3 = -1/8$ and $a_4 = 1/16$, we have $1 <_\delta 3 <_\delta 4 <_\delta 2$. So $\delta[1,4] = 1423$.

Thus $\pi = 1423$ is a subpermutation of δ.



Infinite permutations generated by words

For a word $\omega = \omega_1\omega_2\omega_3\ldots$ over the alphabet $\Sigma = \{0,1\}$ we define the binary real number $R_\omega(i) = 0,\omega_i\omega_{i+1}\ldots = \sum_{k\geq 0}\omega(i+k)2^{-(k+1)}$. Let ω be a right infinite nonperiodic word over the alphabet Σ.

Definition

The *infinite permutation* generated by the word ω is the ordered triple $\delta_\omega = \langle \mathbb{N}, <_{\delta_\omega}, < \rangle$, where $<_{\delta_\omega}$ and $<$ are linear orders on $\mathbb{N}$. The order $<_{\delta_\omega}$ is defined as follows: $i <_{\delta_\omega} j$ if and only if $R_\omega(i) < R_\omega(j)$.

Definition

The *permutation complexity* $\lambda(n)$ of the word ω is the number of distinct subpermutations of δ_ω of length n.

Lemma (Makarov, 2006)

Let ω be a binary nonperoidic infinite word. Then $\lambda(n) \geq C(n-1)$.

Natural question: which words satisfy equality $\lambda(n) = C(n-1)$?

Sturmian words

Theorem (Morse and Hedlund, 1940)

Let ω be an infinite word. Then $C(n) = const$ for any $n \geq N$ iff ω is a periodic word. If ω is a nonperiodic word, then $C(n) \geq n + 1$ for any n.

Definition

An infinite word ω is called a Sturmian word if $C(n) = n + 1$ for any n.

Theorem (Makarov, 2009)

Let ω be a Sturmian word. Then $\lambda(n) = n - 1$ for any n.

So, for the Sturmian words we have $\lambda(n) = C(n - 1)$.

Thue-Morse word

Let $\varphi(0) = 01, \varphi(1) = 10$. We have $\varphi^2(0) = 0110$, $\varphi^3(0) = 01101001$. Then $\lim_{n\to\infty} \varphi^n(0) = 0110100110010110\ldots$ is the Thue-Morse word.

Theorem (Brlek, A.de Luca and Varricchio, 1989)

Let ω be the Thue-Morse word. Then $C(n) = 2n + 2^{k+1} - 2$ for $3 \cdot 2^{k-1} + 1 \leq n < 2^{k+1} + 1$ and $C(n) = 4n - 2^k - 4$ for $2^k + 1 \leq n < 3 \cdot 2^{k+1} + 1$.

Theorem (Widmer, 2011)

Let ω be the Thue-Morse word. Then for any $n \geq 6$, where $n = 2^a + b$ with $0 < b \leq 2^a$, we have $\lambda(n) = 2(2^{a+1} + b - 2)$.

The main theorem

Definition

A word ω is called uniformly recurrent if for any $n > 0$ there exists a number $t_\omega(n)$ such that every $t_\omega(n)$–length factor of ω contains all factors of ω of length n.

Theorem

Let ω be an infinite uniformly recurrent word. Then $\lambda(n) = C(n-1)$ if and only if ω is a Sturmian word.