About vertices of degree 6 of C_3-critical minimal 6-connected graph

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- $V(G)$ — the set of vertices of the graph G;
- $v(G) = |V(G)|$;
- $E(G)$ — the set of edges of the graph G;
- $e(G) = |E(G)|$;
- $d(v)$ — degree of a vertex v;
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Definition

$\kappa(G)$ — the minimal number of vertices: if we remove them from G then we obtain a disconnected or trivial graph (vertex connectivity of G).
Graph G is called k-connected iff $\kappa(G) \geq k$.

Some more notions for k-connected graph

Let $\kappa(G) = k$

- $V_k = \{v \in V(G) \mid d(v) = k\}$;
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This talk is devoted to 6-connected graphs.

Minimal and contraction critical k-connected graphs

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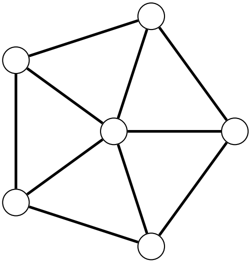

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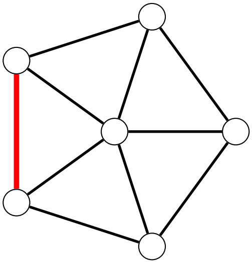

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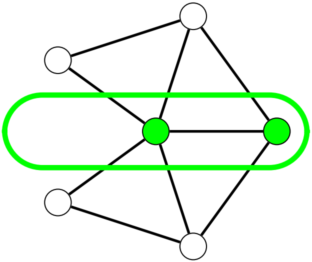

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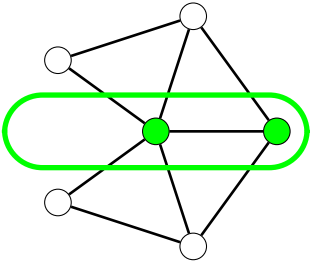

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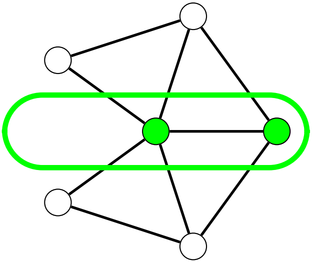

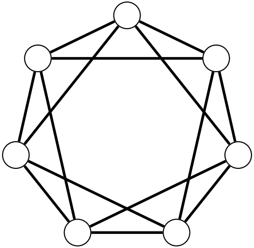

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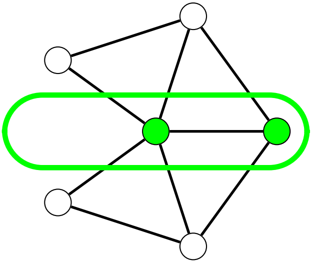

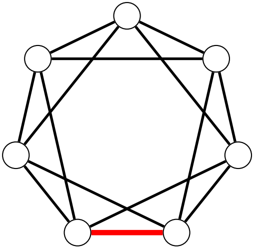

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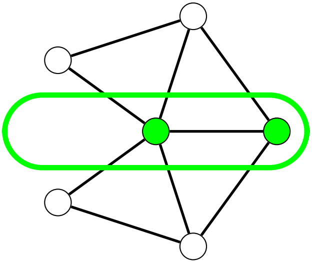

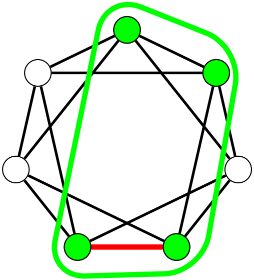

Some results on the vertices of degree k in minimal and contraction critical k-connected graphs

Let G is minimal k-connected graph

- The subgraph G_{k+1} is a forest (W. Mader, 1972).
- $v_k(G) \geq \frac{(k-1)v(G)+2k}{2k-1}$ and this bound is sharp (W. Mader, 1979).

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- For $6 \leq k \leq 10$ we have $v_k(G) \geq \frac{1}{2}v(G)$ and there are infinite series of non-regular examples of such graphs (S.Obraztsova, A.P., 2010-2011).

Mader's definition of C_m-critical k-connected graphs

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k-connected graph G is said to be C_m-critical iff for all $\ell \leq m$ any clique on ℓ vertices is contained in k-cutset of G.

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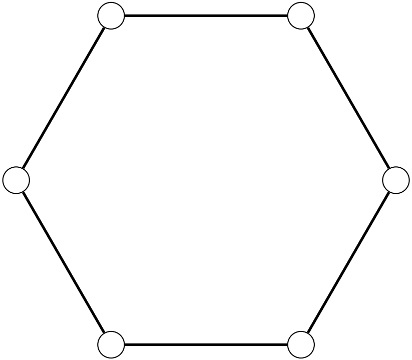
Theorem (W. Mader, 1988)

Any C_3-critical graph is 6-connected.

So that the main point of our interest in this talk is vertices of degree 6 of a C_3-critical 6-connected graph.

An example of non-regular C_3-critical 6-connected graph



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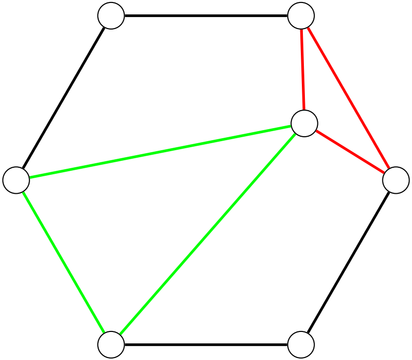

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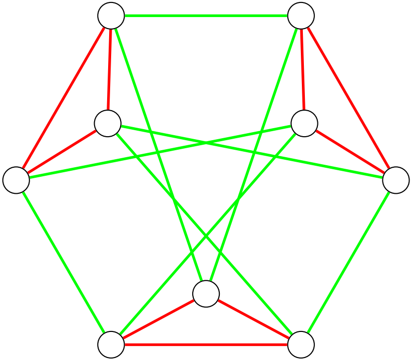

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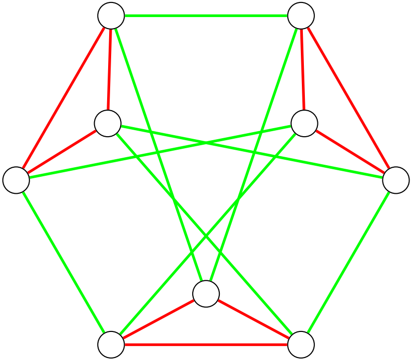

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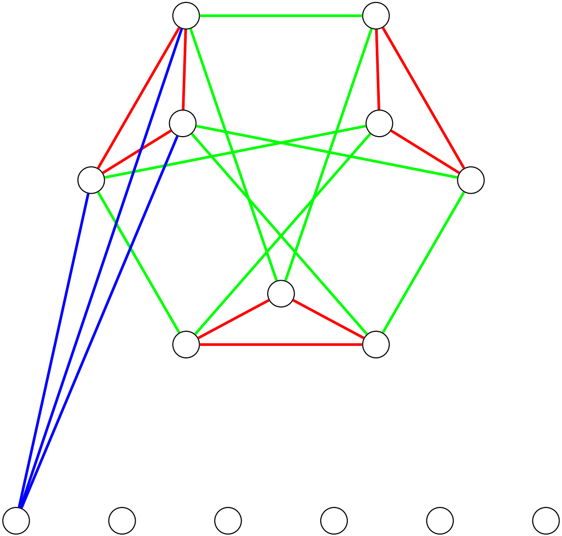

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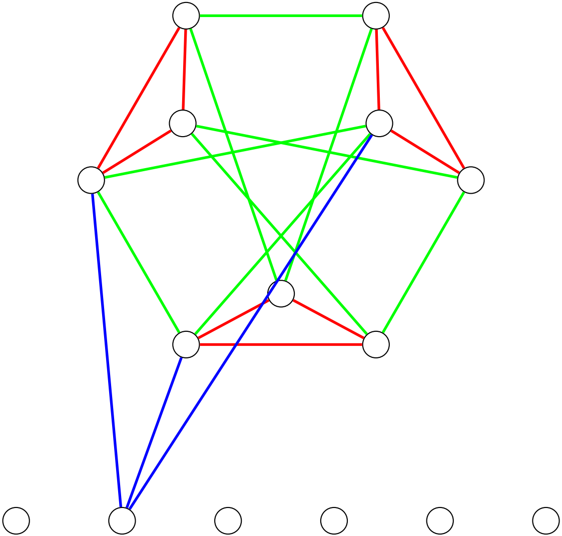

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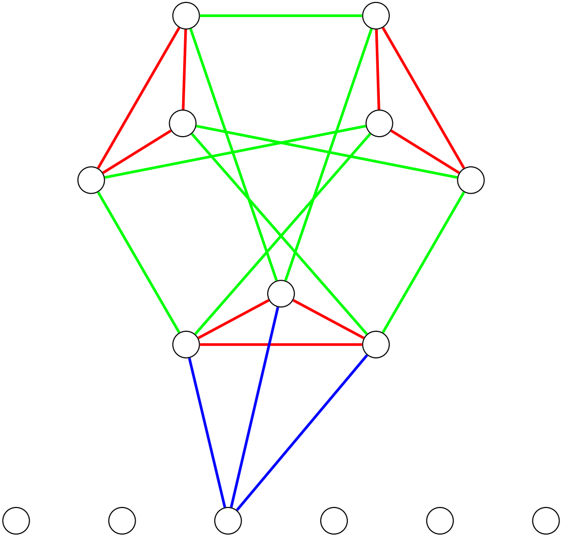

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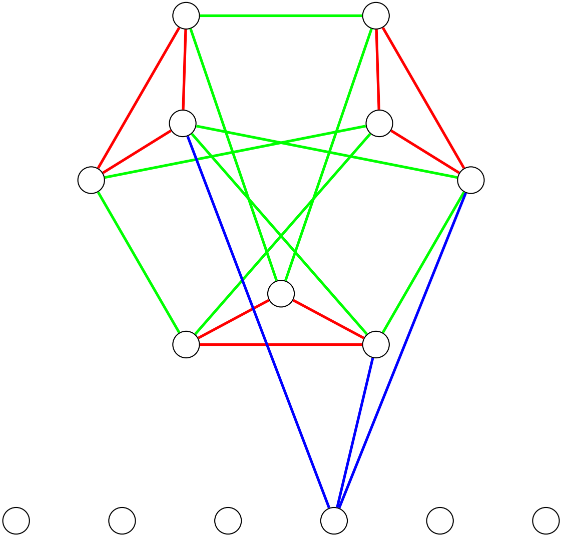

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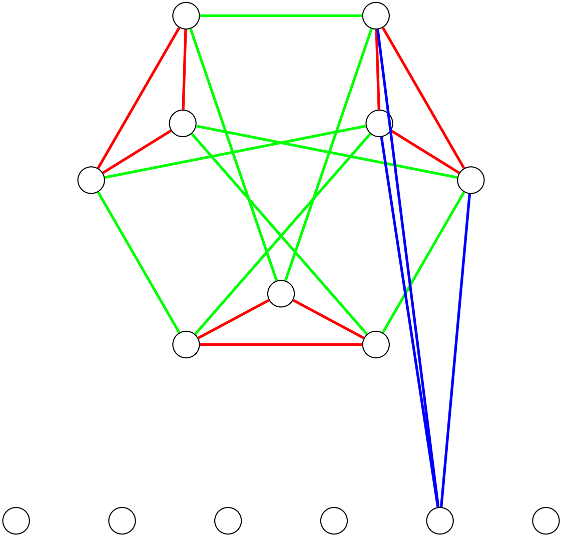

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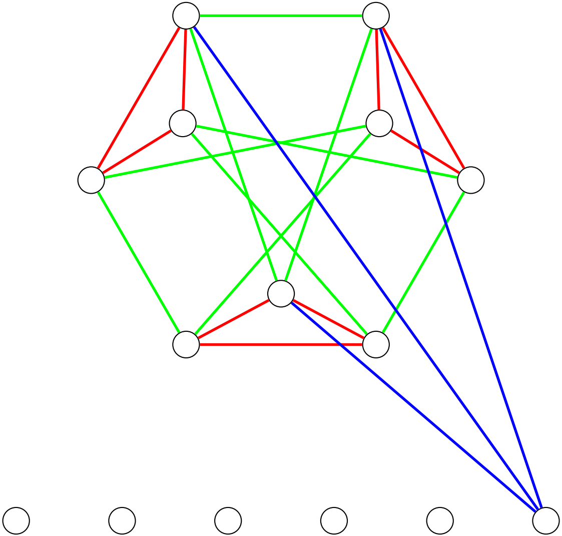

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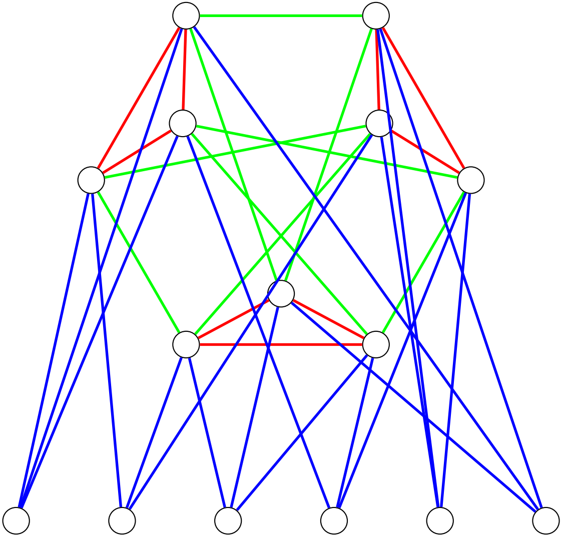

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It is enough to prove that any connected component of G_6 contains a cycle. It's obvious for the components in which all vertices have degree at least 2. So that it's enough to consider a component A of G_6 that contains a vertex a with $d_6(a) = 1$.

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Lemma (S.Obraztsova, 2010)

Let G is a contraction critical k-connected graph and $a \in V_6$. Then there exists cutset T in G, such that $|T| = k$, $a \in T$, there is vertex $b \in T$ adjacent with a and T separates a component H with at most $\frac{k-1}{2}$ vertices.

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Let T is k-cutset in G, separates a component H with $V(H) = \{u, v\}$, and $x \in T$. Then there exists k-cutset R in G, such that $\{u, v, x\} \subset R$.

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Lemma

Let G is a C_3-critical minimal 6-connected graph, $a \in V_6$, $d_6(a) = 1$ and A is a component of G_6, such that $a \in V(A)$. Then the component A contains a cycle.

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How this lemma helps to prove the theorem?

We have 2 sets: V_6 and V_7 and $e_{6,7}$ edges between them. Than by lemma, we have

$$e_{6,7} = 6v_6 - 2e_6 \leq 4v_6.$$

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So that

$$4v_6(G) \geq 5v_7(G) + 2 = 5v(G) - 5v_6(G) + 2,$$

and finally

$$v_6(G) \geq \frac{5}{9}v(G) + \frac{2}{9} > \frac{5}{9}v(G).$$