On Heawood-type problem for maps with tangencies.

Gleb Nenashev

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Four color theorem

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In graph terminology: every planar graph is four-colorable.

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Ringel problem

Let k be a nonnegative integer. We say that a graph is *k-planar*, if it can be drawn on the plane such that any edge intersects at most k other edges.

What is the minimum number of colors necessary to color all 1-planar graphs?

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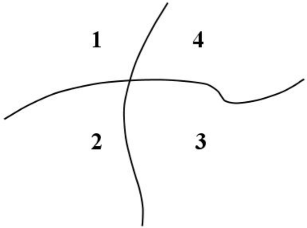
$\chi(G)$ — is the smallest number of colors needed to color the vertices of a graph G such that no two adjacent vertices hame the same color.

$\chi(B)$ — is the smallest number of colors needed to color the regions of map B such that no two regions having a common point share the same color.



Let $\mathcal{C}$ - a class of graphs (maps), we denote by $\chi(\mathcal{C})$ - the minimum number of colors that can be colored every graph (map) in the class $\mathcal{C}$.

The following inequality holds for $g > 0$

$$\chi(\mathcal{B}_{k,g}) \leqslant \frac{2k+1+\sqrt{4k^2-12k+16gk+1}}{2}.$$

Conjecture: The difference between the left and right parts are not greater than some constant c_k.

For $g \geqslant 0$

$$\chi(\mathcal{B}_{k,g}) \leqslant \chi(\mathcal{A}_{\lceil \frac{k-2}{2} \rceil \lfloor \frac{k-2}{2} \rfloor, g}).$$

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For $g > 0$

$$\chi(\mathcal{B}_{4,g}) = \chi(\mathcal{A}_{1,g}) \leqslant \frac{9 + \sqrt{17 + 64g}}{2},$$

$$\chi(\mathcal{B}_{5,g}) \leqslant \chi(\mathcal{A}_{2,g}) \leqslant \frac{11 + \sqrt{41 + 80g}}{2}.$$

- $\mathcal{F}(n) = \frac{n^2-9n+16}{16}$, for n odd.
- $\mathcal{F}(n) = \frac{3n^2-26n+48}{48}$, for n even.

Theorem

$\chi(\mathcal{A}_{1,g})$ is not greater than the maximum n such that $g \geqslant \mathcal{F}(n)$, for $g > 0$.

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$\chi(\mathcal{A}_{1,g}) = \frac{9+\sqrt{17+64g}}{2} - O(log(g))$, for $g > 0$.

Complete graphs K_9, K_{25}, K_{41} and K_{57} can be drawn on the corespondent surfaces such that they give equality in the first theorem.

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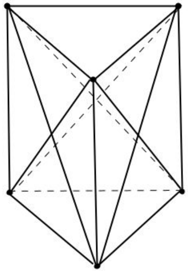
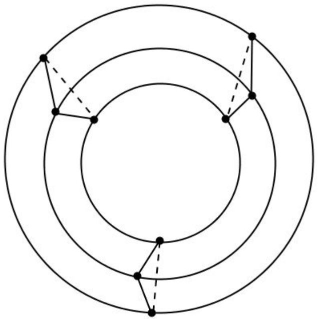
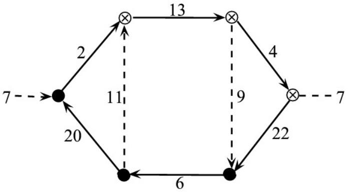
K_9

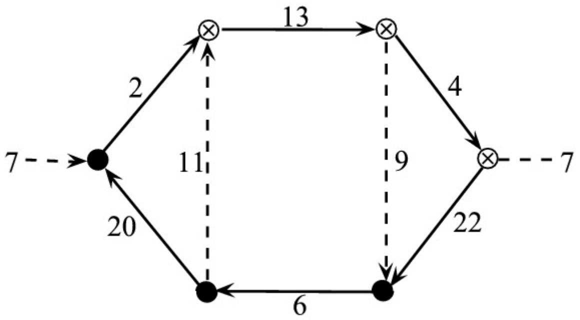




K_{25}, $25 = 9 + 16t$



1. Cubic graph on $4t + 2$ vertices.
2. Bypass: it by turning in each black vertex clockwise, and in white conterclockwise. We obtain a cycle when we go along each edge twice (in both direction).
3. Dashed edges form a perfect matching.
4. We defined the orientation and numbers on the edges such that the sum of incoming numbers and outcoming are equal modulo $9 + 16t$.
5. All $\pm$ number on the edges, and the difference between the numbers in the bypass before and after each dashed edge are different modulo $9 + 16t$.



Bypass: in black vertcies clockwise, in white counterclockwise

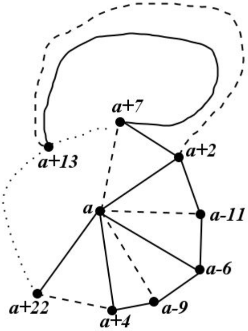
$\mathbf{0}$: (7) 2 (-11) -6 (-9) 4 22 6 20 (-7) -4 -13 -2 -20 (11) 13 (9) -22

$\mathbf{a}$: $(a+7)$ $a+2$ $(a-11)$ $a-6$ $(a-9)$ $a+4$ $a+22\dots$

The ribbon structure of our graph:

$\mathbf{a}:\ (a+7)\ \ a+2\ \ (a-11)\ \ a-6\ \ (a-9)\ \ a+4\ \ a+22\ldots$

$\mathbf{a+7}:\ \ \ldots a+7+6\ \ a+7+20\ \ (a)\ldots$



All faces are triangles and each of them has exactly one dashed edge.

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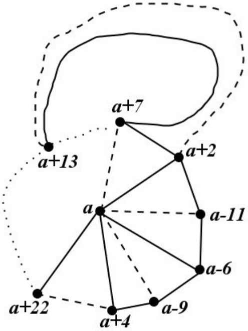

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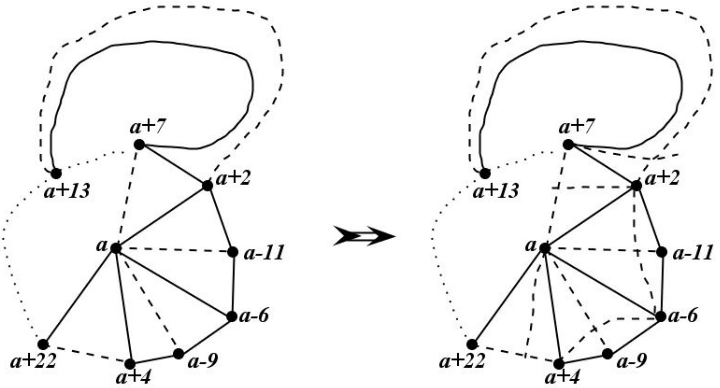

We draw in two adjacent triangles (by dyshed edge) the diagonal edge across the comon dashed edge.
We get a complete graph if our auxiliary cubic graph satisfied those 5 conditions wrotten above.

Thank you for your attention