Principal (left) ideal languages, constants and synchronizing automata

Marina Maslennikova [1] and Emanuele Rodaro [2]

[1] Ural Federal University, Ekaterinburg, Russia

[2] University of Porto, Porto, Portugal

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Institute of Mathematics and Computer Science
Ural Federal University, Ekaterinburg, Russia

Definitions

- A *deterministic finite automaton* (DFA) is a triple $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$. We do not need any initial and final states.
- We often write $q \,.\, w$ for $\delta(q, w)$ and $P \,.\, w$ for $\delta(P, w) = \{\delta(q, w) \mid q \in P\}$.
- A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called *synchronizing* if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is $|Q \,.\, w| = 1$.
- Any word with this property is said to be *reset* for the DFA $\mathscr{A}$.
- $\mathrm{Syn}(\mathscr{A})$ is the language of all reset words for $\mathscr{A}$.
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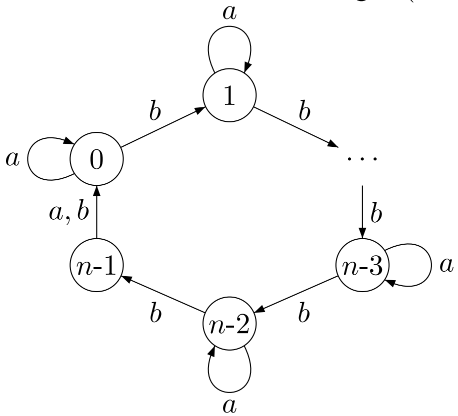
The Černý conjecture

In 1964 Jan Černý found an infinite series of n-state synchronizing automata whose shortest reset word has length $(n-1)^2$.



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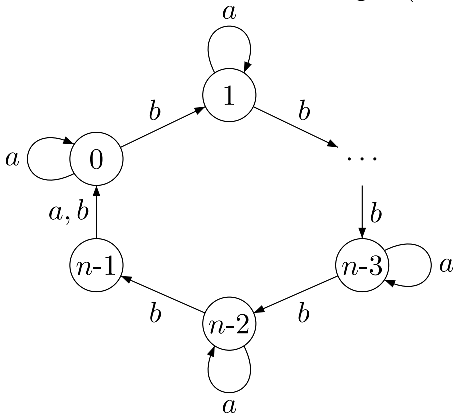

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- A language $I \subseteq \Sigma^*$ is called a *two-sided* (*right* or *left*, respectively) *ideal* if $I \neq \emptyset$ and $I = \Sigma^* I \Sigma^*$ ($I = I\Sigma^*$ or $I = \Sigma^* I$, respectively).
- The *reset complexity* of a two-sided ideal I is the minimal possible number of states in a synchronizing automaton $\mathscr{B}$ such that $\mathrm{Syn}(\mathscr{B}) = I$.

The Černý conjecture (reformulation)

The Černý conjecture holds true iff $rc(I) \geq \sqrt{||I||} + 1$ for every two-sided ideal I.

Ideals and the Černý conjecture

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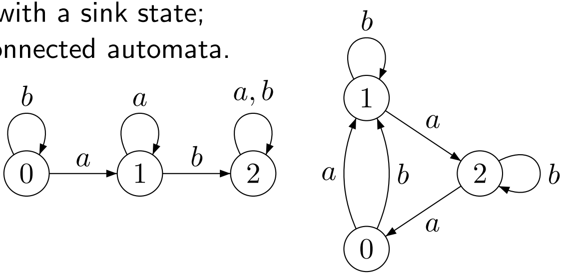

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Given a two-sided ideal I, does there always exist a strongly connected synchronizing automaton $\mathscr{B}$ with $\mathrm{Syn}(\mathscr{B}) = I$?

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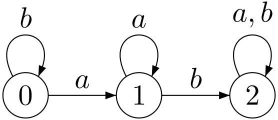

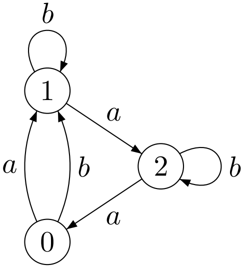

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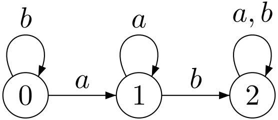

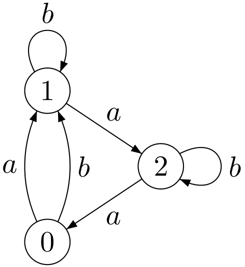

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Let I be a two-sided ideal language over non-unary alphabet. There is a strongly connected DFA $\mathscr{B}$ s.t. $\mathrm{Syn}(\mathscr{B}) = I$.

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If I is a principal two-sided ideal, i.e. $I = \Sigma^* w \Sigma^*$, then there is an algorithm to construct a strongly connected synchronizing automaton $\mathscr{B}$ with $|w| + 1$ states such that $\mathrm{Syn}(\mathscr{B}) = I$.

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- The *(left) quotient* $w^{-1}L$ of a language $L \subseteq \Sigma^*$ by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x | wx \in L\}$.
- A DFA $\mathscr{A} = \langle Q, \Sigma, \delta, q_0, \{q_0\}\rangle$ is called *trim* if each state $q \in Q$ is reachable from q_0 and q_0 is reachable from each state $q \in Q$.
- $\mathcal{L}(\Sigma)$ is the class of all trim automata $\mathscr{A}$ with $L[\mathscr{A}] = w^{-1}\Sigma^* w$ for some $w \in \Sigma^*$.
- A DFA $\mathscr{B} = \langle Q_2, \Sigma, \delta_2\rangle$ is a *homomorphic image* of a DFA $\mathscr{A} = \langle Q_1, \Sigma, \delta_1\rangle$ if there is a map $\varphi : Q_1 \to Q_2$ preserving the action of letters.
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Lemma 1.

Let $\mathscr{A}$ be a trim DFA such that $L[\mathscr{A}] = w^{-1}\Sigma^* w$ for some $w \in \Sigma^*$. Hence $\mathscr{A}$ is a strongly connected synchronizing automaton with $w \in \mathrm{Syn}(\mathscr{A})$.

Theorem 2.

Let $\mathscr{B} = \langle Q, \Sigma, \delta \rangle$ be a strongly connected synchronizing automaton. For each reset word $w \in \mathrm{Syn}(\mathscr{B})$ of minimal length there is a DFA $\mathscr{A} \in \mathcal{L}(\Sigma)$ with $L[\mathscr{A}] = w^{-1}\Sigma^* w$ and

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Lifting property

- Strongly connected synchronizing automata are all and only homomorphic images of trim automata recognizing languages of the kind $w^{-1}\Sigma^* w$.
- Cong_k is the (maybe empty) set of all congruences of an automaton of index k.

Theorem 3.

Cerny's conjecture holds if and only if for any $\mathscr{B} \in \mathcal{L}(\Sigma)$ and $\rho \in \mathrm{Cong}_k(\mathscr{B})$ for all $k < \sqrt{\|\mathrm{Syn}(\mathscr{B})\|} + 1$ we have

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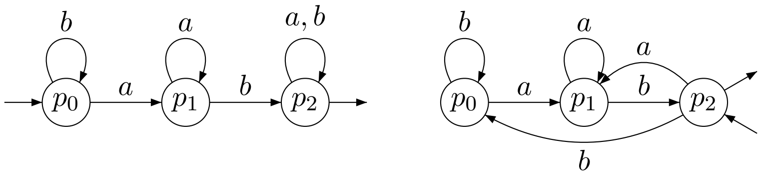
The construction is similar to the construction of the minimal DFA recognizing the language $L = \Sigma^* w \Sigma^*$.



The states are enumerated by prefixes of w: $p_0 = \varepsilon$, $p_1 = a$, $p_2 = ab$.

The initial (and also final) state is w.

$p_i \,.\, a = p_j$ iff p_j is the maximal suffix of $p_i a$ that appears in w as a prefix.

- The minimal automaton $\mathscr{A}_w$ recognizing $w^{-1}\Sigma^* w$ is synchronizing and $L[\mathscr{A}_w] \cap \mathrm{Syn}(\mathscr{A}_w) \neq \emptyset$, since $w \in L[\mathscr{A}_w] \cap \mathrm{Syn}(\mathscr{A}_w)$.
- Question: how to describe regular languages whose minimal recognizing automaton is synchronizing?
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Constants

- Let $L \subseteq \Sigma^*$ be a regular language. A word $w \in \Sigma^*$ is a *constant* for L if the implication

$$u_1 w u_2 \in L, u_3 w u_4 \in L \Rightarrow u_1 w u_4 \in L$$

  holds for all $u_1, u_2, u_3, u_4 \in \Sigma^*$.
- $C(L)$ is the set of all constants of a regular language L.
- $Z(L) = \{w | \Sigma^* w \Sigma^* \cap L = \emptyset\}$, $Z(L) \subseteq C(L)$.
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Theorem 4.

The minimal automaton $\mathscr{A}$ recognizing a language L is synchronizing and $L \cap \mathrm{Syn}(\mathscr{A}) \neq \emptyset$ if and only if the following properties hold:
(i) $C(L) \neq \emptyset$;
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The conditions (i) and (ii) can be checked in polynomial of the size of $\mathscr{A}$ time.

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Conclusion

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- Strongly connected synchronizing automata are all and only homomorphic images of trim automata recognizing languages of the kind $w^{-1}\Sigma^* w$.
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Conclusion

- For every two-sided ideal language I over non-unary alphabet there is some synchronizing DFA $\mathscr{B}$ such that $\mathrm{Syn}(\mathscr{B}) = I$.
- Strongly connected synchronizing automata are all and only homomorphic images of trim automata recognizing languages of the kind $w^{-1}\Sigma^* w$.
- The criterion for the minimal automaton recognizing a regular language L to be synchronized by some word from L can be stated in terms of constants of L.