Descriptive complexity of additive shift of regular language

RuFiDiM, Petrozavodsk, 15 September 2014

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Definitions

A deterministic finite automaton $A = \langle Q, \Sigma, \delta, q_0, T \rangle$ is defined by specifying
Q – a finite set of states
Σ – finite alphabet
$\delta\colon Q \times \Sigma \to Q$ – transition function
$q_0 \in Q$ – initial state
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$\Lambda = \{\langle Q, \Sigma, \delta, q_0, T\rangle | \Sigma = \{0, 1\}\}$

Definitions

$bin\colon \mathbb{N} \to \Sigma^*$

$bin(x) = a_{k-1}a_{k-2}\dots a_0 \iff$

$x = 2^{k-1}a_{k-1} + 2^{k-2}a_{k-2} + \cdots + a_0$, $a_i \in \{0,1\}$, $a_{k-1} = 1$

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$\forall A \in \Lambda, x \in \mathbb{N}\ x \in L(A) \iff bin(x) \in L(A)$

Problem

Given $A \in \Lambda$, $k \in \mathbb{N}$, does such $B \in \Lambda$ exists that $\forall x \in \mathbb{N}\ x \in L(B) \iff (x + k) \in L(A)$?

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If it does, how large is size of B?

Main result

$\forall A \in \Lambda \forall k \in \mathbb{N}$
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$|Q_B| \leqslant \left\lceil \frac{k}{2} \right\rceil n^2 + 2nk + 2k$, where $n = |Q_A|$.

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– Generalize used construction for $k > 1$ but with larger number of states than required

– Prove that built automaton after minimization has required number of states

k = 1

$10001 + 1 = 10010$

$\mathbf{k = 1}$

$10001 + 1 = 10010$

$100010 + 1 = 100011$

$100011 + 1 = 100100$

$\mathbf{k = 1}$

$$x = 2x' + x'', \; x'' \in \{0, 1\}$$
$$A = \langle Q, \Sigma, \delta, q_0, T \rangle \in \Lambda$$

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$A = \langle Q, \Sigma, \delta, q_0, T \rangle \in \Lambda$

$\delta(q_0, bin(x + 1)) = \delta(\delta(q_0, bin(x')), 1)$ if $x'' = 0$,
$\delta(q_0, bin(x + 1)) = \delta(\delta(q_0, bin(x' + 1)), 0)$ if $x'' = 1$

Solution for k = 1

$A = \langle Q, \Sigma, q_0, \delta, T \rangle$, $B = \langle Q', \Sigma, q_0', \delta', T' \rangle$.

$$
\begin{aligned}
Q' &= Q \times Q \\
q_0' &= (q_0, \delta(q_0, 1)) \\
\delta'((a,b),0) &= (\delta(a,0), \delta(a,1)) \\
\delta'((a,b),1) &= (\delta(a,1), \delta(b,0)) \\
T' &= Q \times T
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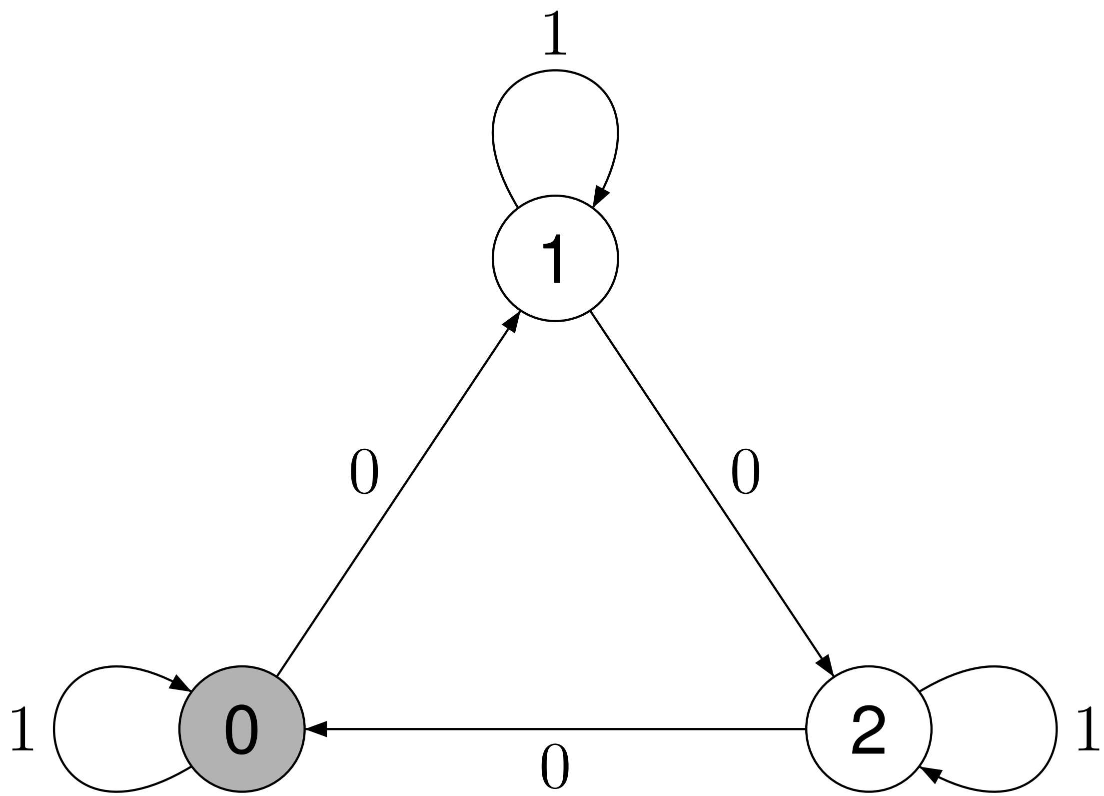
$$\forall x \in \mathbb{N}\ \delta'(q_0', bin(x)) = (\delta(q_0, bin(x)), \delta(q_0, bin(x+1)))$$

Example



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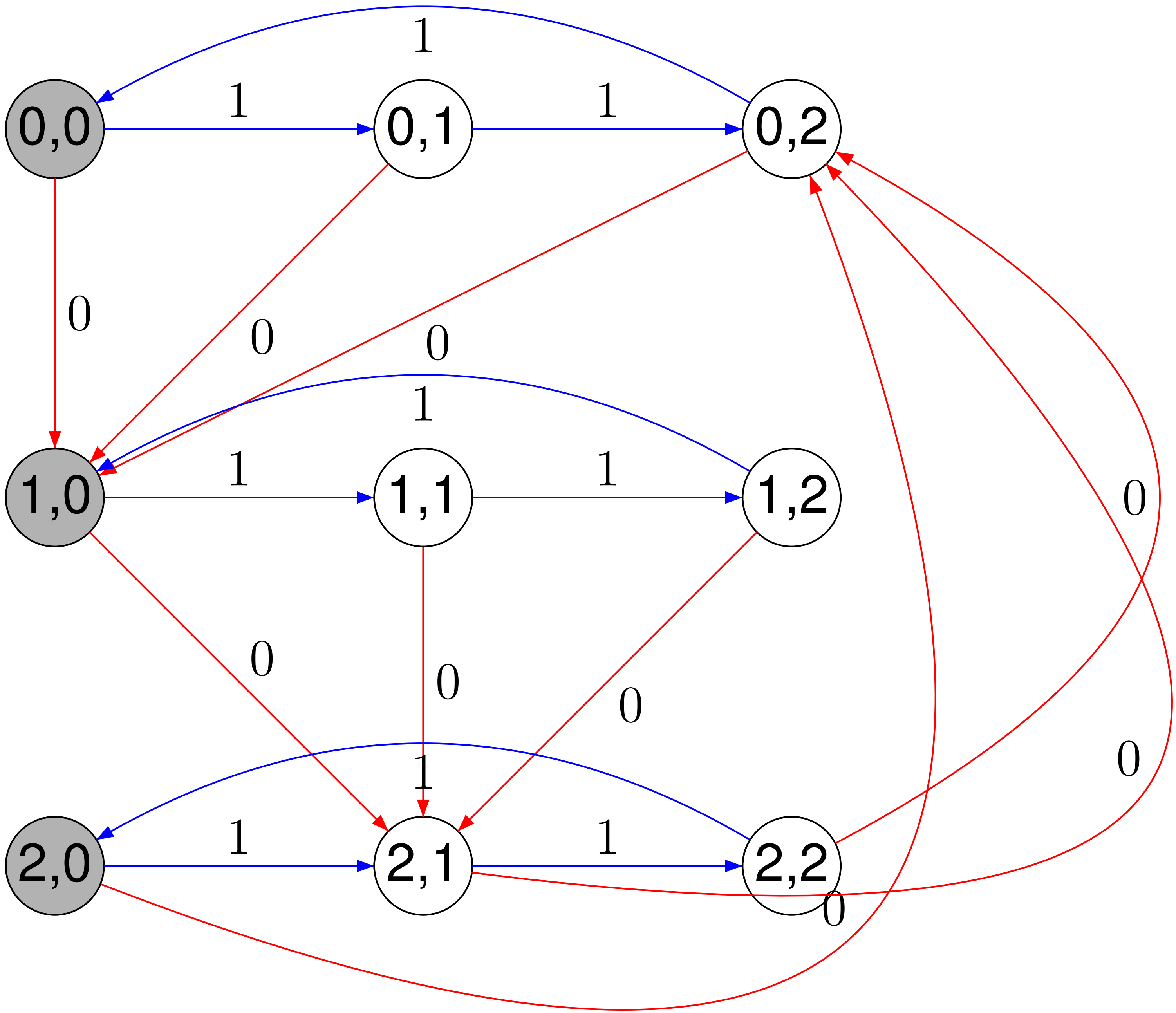

$\mathbf{k > 1}$

$10001 + 1 = 10010$

k > 1

$10001 + 1 = 10010$

$10001001 + 101 = 10001110$

$10001110 + 101 = 10010011$

Solution for k > 1

$A = \langle Q, \Sigma, q_0, \delta, T \rangle$, $B = \langle Q', \Sigma, q'_0, \delta', T' \rangle$

$m = 2^{\lceil \log_2 k \rceil}$

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$Q \times Q \times \{0 \ldots m-1\} \cup \{0 \ldots m-1\} q_0' = 0$

$$\delta'(q, a) = \begin{cases} 2q + a, & 2q + a < m \\ (\delta(q_0, 1), \delta(q_0, 10), (2q + a - m)), & 2q + a \geqslant m \end{cases}$$

$$\delta'((p_1, p_2, q), a) = \begin{cases} (\delta(p_1, 0), \delta(p_1, 1), (2q + a)), & 2q + a < m \\ (\delta(p_1, 1), \delta(p_2, 0), (2q + a - m)), & 2q + a \geqslant m \end{cases}$$

Solution for k > 1

$$A = \langle Q, \Sigma, q_0, \delta, T\rangle,\ B = \langle Q', \Sigma, q_0', \delta', T'\rangle$$
$$m = 2^{\lceil \log_2 k \rceil}$$
$$Q \times Q \times \{0 \dots m-1\} \cup \{0 \dots m-1\} q_0' = 0$$
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$$\begin{aligned} T' = \{&x \in \{0 \dots m-1\} | (x + k) \in L(A)\} \\ &\cup \{(a, b, q) | q + k < m, \delta(a, bin(q + k)) \in T\} \\ &\cup \{(a, b, q) | q + k \geqslant m, \delta(b, bin(q + k - m))\} \end{aligned}$$

Solution for k > 1

For any number $x \in \mathbb{N}$
$\delta'(q'_0, x) = x$ if $x < m$
$\delta'(q'_0, x) = (\delta(q_0, bin(\left\lfloor \frac{x}{m} \right\rfloor)), \delta(q_0, bin(\left\lfloor \frac{x}{m} \right\rfloor + 1)), \mathrm{mod}_m x)$
otherwise.

End

Thank you!